Indian Statistical Institute, Bangalore
M. Math. First Year
First Semester - Linear Algebra
Semestral Exam Duration : 3 hours Max Marks 100 Date : November 22, 2017

1. Let V be a finite dimensional vector space and $S, T : V \to V$ be linear operators.
   (a) If all the subspaces of V are T-invariant then show that T is a scalar operator (i.e., multiplication by a fixed scalar).
   (b) If S and T commute then show that all the eigenspaces of S are T-invariant.
   (c) Recall that S is called an idempotent on V if $S^2 = S$. If T commutes with all the idempotents on V, then use the previous parts to show that T is a scalar operator. [10 + 5 + 10 = 25]

2. (a) If $\{x_1, ..., x_n\}$ is a basis of a vector space V and $0 \leq k \leq n$, then show that the set $\{x_{i_1} \wedge x_{i_2} \wedge ... \wedge x_{i_k} : 1 \leq i_1 < i_2 ... < i_k \leq n\}$ is a basis of the kth wedge power of V.
   (b) Prove that the $(n-k)$th wedge power of V is naturally isomorphic to the dual of the kth wedge power. [12 + 13 = 25]

3. Let $T : V \to V$ be a linear operator on a finite dimensional vector space.
   (a) Define the determinant of T in terms of the wedge powers of T. Show that it is well defined.
   (b) Prove that, in terms of the matrix of T (wrt a given basis) the determinant of T is given by a Cauchy expansion.
   (c) Using the definition in part(a), show that if S is another linear operator on V then det (ST) = det (S) det (T). [8 + 9 + 8 = 25]

4. Let R be a Noetherian (commutative unital) ring.
   (a) If M is any finitely generated module over R then show that all the submodules of M are finitely generated.
   (b) If R is an euclidean domain then show that all submodules of the R-module R^n are free. [10 + 15 = 25]